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March, 1973

NIC # 15334
Packet Radio Temporary Note # 37

CARRIER-SENSE WITH INITIAL RANDOM TRANSMISSION DELAY

1. Introduction

In packet radio note #24, we analyzed the throughput of carrier sense according to a scheme in which a terminal, with a packet ready for transmission, transmits the packet as soon as it senses the channel is free. Thus, if two or more packets accumulate during a transmission period, then, at the next transmission period, a conflict will occur with probability one.

The idea of randomizing the starting times of transmission of the accumulated packets then seems attractive for interference reduction and throughput improvement.

2. The Scheme Under Study

More precisely, the system under study here corresponds to the following:

The system is slotted and entirely synchronized.

The slot size is equal to τ , the propagation delay.

Consider a terminal with a packet ready for transmission. If the channel is sensed to be idle, then

- with probability p , the terminal transmits the packet.
- With probability $1 - p$, the terminal delays the transmission of the packet by τ seconds (i.e., one slot).

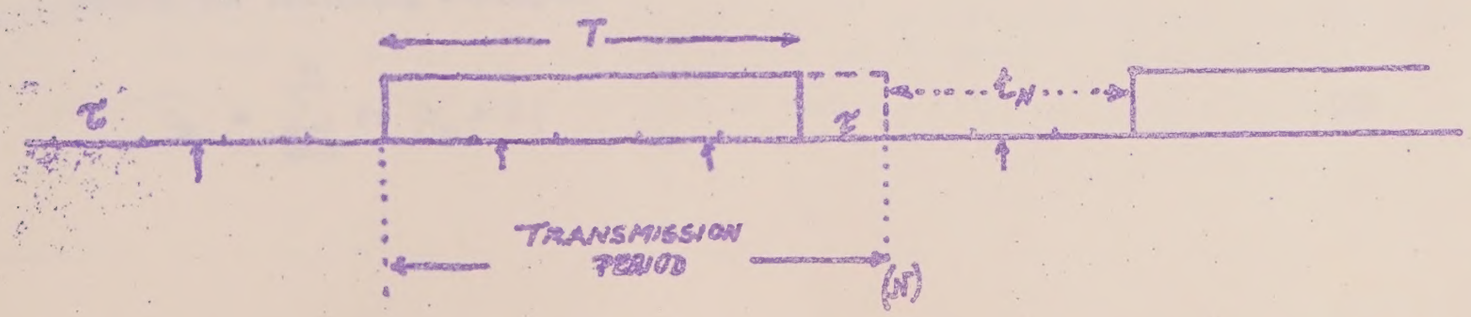
If, at this new point in time, the channel is still detected to be idle, the same process described above is repeated; otherwise, a retransmission delay large compared to the packet transmission time is assumed to be incurred.

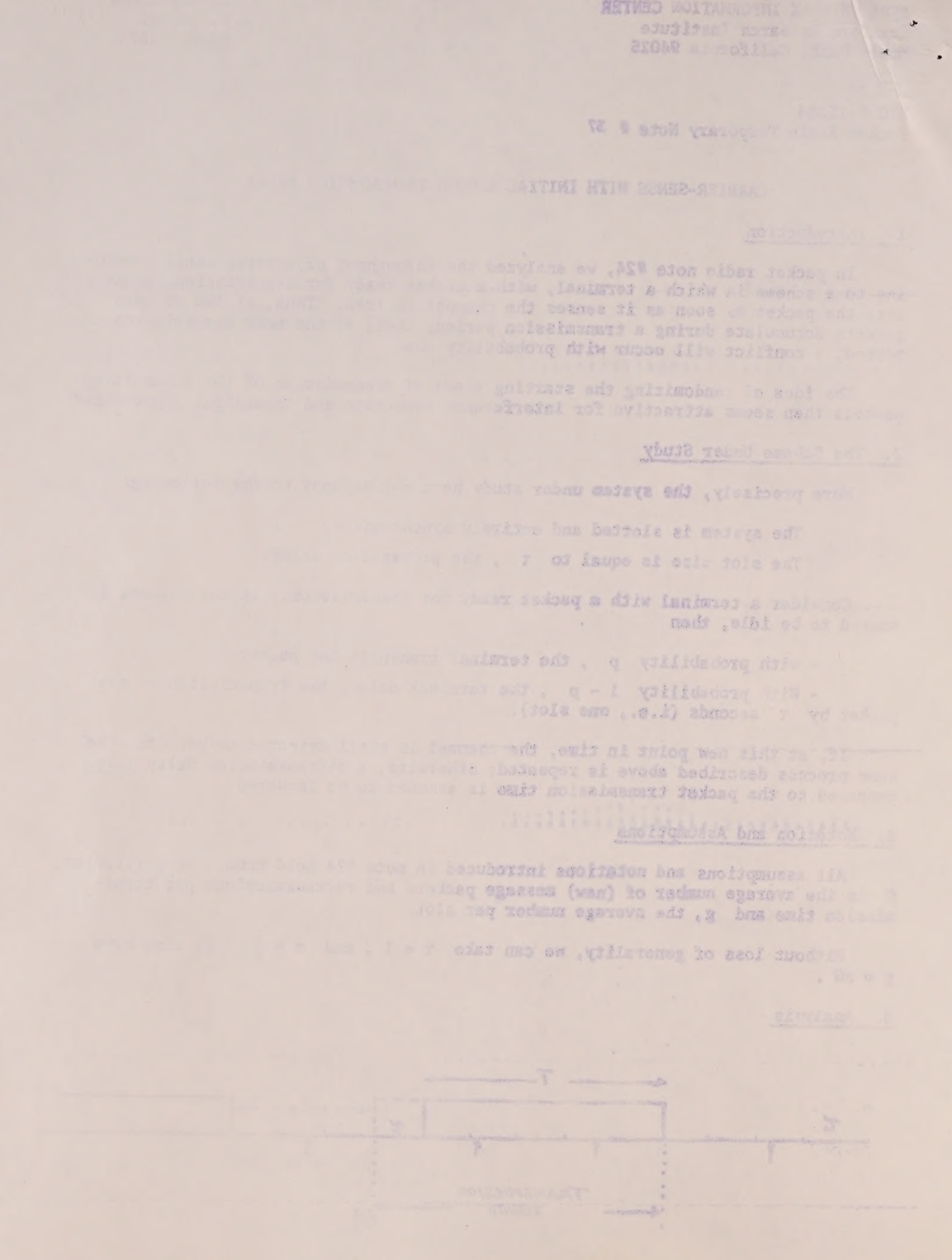
3. Notation and Assumptions

All assumptions and notations introduced in note #24 hold true. In particular, G is the average number of (new) message packets and retransmissions per transmission time and g, the average number per slot.

Without loss of generality, we can take $\tau = 1$, and $a = \frac{1}{T}$. We also have $g = aG$.

4. Analysis





4.1 Distribution of Number of Packets Accumulated at End of Transmission Period

Let N denote this number. According to the scheme described in Section 2, only those packets arriving during a transmission period will accumulate at the end of that transmission period. Therefore

$$P_N = \Pr\{N \text{ packets accumulate at end of transmission period}\}$$

$$= \frac{[(T+1)g]^N}{N!} e^{-(T+1)g} \tag{1}$$

4.2 Distribution of Idle Time Between Two Transmission Periods

Given N , let t_N be the number of slots elapsed until some packet is transmitted. Let $q = 1-p$. It is easy to see that

$$\Pr\{t_N > k\} = (1-p)^{(k+1)N} \prod_{j=1}^{k} \left[\sum_{n=0}^{\infty} e^{-g} \frac{g^n}{n!} (1-p)^{n(k-j+1)} \right]$$

$$= q^{(k+1)N} \prod_{j=1}^{k} e^{g[q^{k-j+1} - 1]}$$

$$= q^{(k+1)N} \; e^{g\left(\frac{q(1-q^k)}{p} - k\right)} \tag{2}$$

and therefore, for $k > 0$ we have

$$\Pr\{t_N = k\} = \Pr\{t_N > k-1\} - \Pr\{t_N > k\}$$

$$= q^{kN} \cdot e^{g\left[\frac{q(1-q^{k-1})}{p} - (k-1)\right]} \cdot \left[1 - q^N e^{-g(1-q^k)}\right] \tag{3}$$

and for $k = 0$:

$$\Pr\{t_N = 0\} = 1 - q^N \tag{4}$$

Note that a conflict will certainly occur if two or more terminals decide to start transmission at the same slot and that there can be only one successful transmission during a transmission period.

Define the following averages:

$$\bar{t}_N = \sum_{k=0}^{\infty} \Pr\{t_N > k\} \tag{5}$$

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$$P_N^N = \Pr\{N \text{ packets accumulate at end of transmission period}\}$$

$$= \frac{[(T+1)g]_N}{N!} e^{-(T+1)g} \qquad (1)$$

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$$\Pr\{t^N > k\} = (1-p)^{(k+1)N} \left[\prod_{j=1}^{k} \sum_{n=0}^{\infty} e^{-g} \frac{g^n}{n!} (1-p)^{n(k-j+1)} \right]$$

$$= q^{(k+1)N} \prod_{j=1}^{k} e^{g\left[q^{(k-j+1)} - 1\right]}$$

$$= q^{(k+1)N} \cdot e^{g\left(\frac{q(1-q^k)}{p} - k\right)} \qquad (3)$$

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Note that a conflict will certainly occur if two or more terminals decide to start transmission at the same slot and that there can be only one successful transmission during a transmission period.

Define the following averages:

$$\bar{t}^N = \sum_{k=0}^{\infty} \Pr\{t^N > k\} \qquad (6)$$

$$\bar{t} = \sum_{N=1}^{\infty} \bar{t}_N \frac{P_N}{1 - P_0} \tag{6}$$

4.3 Probability of Success Over a Transmission Period

Consider a transmission period $\mathcal{T}$. Given N, define

$P_s(N)$: the probability of success over $\mathcal{T}$

$L_N + 1$: the number of packets present at the starting time of $\mathcal{T}$

We have

$$\Pr\{L_N = \ell / t_N = k\} = \frac{(kg)^{\ell-N+1}}{(\ell-N+1)!} e^{-kg} \qquad \ell \geq N-1 \tag{7}$$

$$\Pr\{L_N = \ell\} = \sum_{k=1}^{\infty} \frac{(kg)^{\ell-N+1}}{(\ell-N+1)!} e^{-kg} \Pr\{t_N = k\}$$

$$+ (1-q^N)\, \delta_{\ell,N-1} \qquad \ell \geq N-1 \tag{8}$$

where $\delta_{i,j}$ is the Kronecker symbol and

$$P_s(N) = \sum_{\ell=N-1}^{\infty} q^{\ell} \Pr\{L_N = \ell\} \tag{9}$$

Removing the condition on N, we get

$$P_s = \sum_{N=1}^{\infty} P_s(N) \frac{P_N}{1 - P_0} \tag{10}$$

4.4 Throughput of the Channel

We shall apply here the cycle analysis presented in Note #24 (section IV.1).

It is clear that the number of transmission periods in a busy period is equal to m with probability $P_0(1-P_0)^{m-1}$.

Consider a busy period with m transmission periods. Let N_i denote the number of packets accumulated at the end of the i^{th} transmission period. We know that $N_m = 0$, and that all other N_i are independent and identically distributed random variables. Conditioned on the fact we know the sequence $\{N_i\}$, i=1 to m-1, the average busy period is given by

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$$\bar{B}_m\left(\{N_i\}\right) = \bar{t}_1 + \sum_{i=1}^{m-1} \bar{t}_{N_i} + m(T+1) \tag{11}$$

and the expected time, during the busy period, that the channel is used without conflicts is given by:

$$\bar{U}_m\left(\{N_i\}\right) = T \cdot P_s(1) + \sum_{i=1}^{m-1} T \cdot P_s(N_i) \tag{12}$$

On the other hand, we know that

$$\Pr\{N_i = n\} = \frac{P_n}{1 - P_0} \qquad n \geq 1 \quad i = 1, \ldots, m-1 \tag{13}$$

Therefore, removing the last condition, we get

$$\bar{B}_m = \bar{t}_1 + (m-1)\bar{t} + m(T+1) \tag{14}$$

$$\bar{U}_m = T\,P_s(1) + (m-1)\,T.P_s \tag{15}$$

and removing the condition on m we get

$$\bar{B} = \sum_{m=1}^{\infty} \bar{B}_m P_0 (1-P_0)^{m-1} = \bar{t}_1 + \frac{\bar{t}(1-P_0) + T + 1}{P_0} \tag{16}$$

$$\bar{U} = T\left[P_s(1) + \frac{1 - P_0}{P_0} P_s\right] \tag{17}$$

and the average idle period

$$\bar{I} = 1/g \tag{18}$$

The throughput is a function of G, p and a, and it is expressed as

$$S(G,p,a) = \frac{T\left[P_s(1) + \frac{1-P_0}{P_0} P_s\right]}{\bar{t}_1 + \bar{t}\frac{(1-P_0)}{P_0} + \frac{T+1}{P_0} + \frac{1}{g}}$$

$$= \frac{G\left[P_s(1)P_0 + P_s(1-P_0)\right]}{aG\bar{t}_1 P_0 + G(1+a) + aG\bar{t}(1-P_0) + P_0} \tag{19}$$

$$\bar{\varepsilon}^m \left(N_1^1\right) = \bar{\varepsilon}^1 + \sum_{i=1}^{m-1} \bar{\varepsilon}^{N_i^1} + m(1+\bar{\varepsilon}) \tag{11}$$

and the expected time, during the busy period, that the channel is used without conflicts is given by:

$$\bar{U}^m \left(N_1^1\right) = 1 \cdot p^s(1) + \sum_{i=1}^{m-1} 1 \cdot p^s(N_i^1) \tag{12}$$

On the other hand, we know that

$$\Pr\left\{N_i^1 = n\right\} = \frac{1 - p^0}{p^n} \qquad n \geq 1 \quad i = 1, \ldots, m-1 \tag{13}$$

Therefore, removing the last condition, we get

$$\bar{\varepsilon}^m = \bar{\varepsilon}^1 + (m-1)\bar{\varepsilon} + m(1+\bar{\varepsilon}) \tag{14}$$

$$\bar{U}^m = 1\, p^s(1) + (m-1)\, 1 \cdot p^s \tag{15}$$

and removing the condition on m we get

$$\bar{B} = \sum_{m=1}^{\infty} B^m\, p^0\, (1-p^0)^{m-1} = \bar{\varepsilon}^1 + \frac{\bar{\varepsilon}(1-p^0) + 1 + \bar{\varepsilon}}{p^0} \tag{16}$$

$$\bar{B} = 1\left[p^s(1) + \frac{p^0}{1 - p^0} p^s\right] \tag{17}$$

and the average idle period

$$\bar{I} = 1/g \tag{18}$$

The throughput is a function of G, p and a, and it is expressed as

$$s(G,p,a) = \frac{1\left[p^s(1) + \frac{p^0}{1-p^0} p^s\right]}{\bar{\varepsilon}^1 + \bar{\varepsilon}\frac{p^0}{(1-p^0)} + \frac{p^0}{1+\bar{\varepsilon}} + \frac{a}{1}} \tag{19}$$

$$= \frac{a\bar{\varepsilon}^1 p^0 + G(1+a) + a\bar{\varepsilon}(1-p^0) + p^0}{G\left[p^s(1)p^0 + p^s(1-p^0)\right]}$$

5. Numerical Results and Concluding Remarks

In order to get the throughput of the channel, a PL/I program was written and run on the IBM 360/91 at CCN. Figures 1, 2, and 3 display the throughput as a function of G (channel traffic) for different values of $a = \frac{\tau}{T}$ (0.01, 0.05, and 0.1, respectively) and different values of p (0.1 - 0.99).

For smaller values of p ($0.01 < p < 0.1$), the numerical computations become time consuming and require an extremely large amount of storage. Fortunately, some approximations are possible and the reader can see, in the appendix, that these approximations can lead to a closed form solution for the throughput and also can give accurate results as shown in Table 1.

Figure 4 shows the maximum throughput as a function of p , for $a = 0.01$, 0.05, and 0.1).

Figure 5 shows the ultimate performance (throughput maximized over G and p) as a function of a . For the useful range of a ($0 < a < 0.1$), it is clear that the throughput in the present system is superior to all those considered previously in Packet Radio Note #24 (i.e., pure aloha, slotted aloha, carrier-sense).

6. Delay Considerations

It has been established in Packet Radio Note #7 that the hop-by-hop acknowledgment scheme in terms of delay or throughput is superior to the end-to-end acknowledgment scheme. In this section the intent is to bring the delay into picture and compare the delay-throughput trade-off in the following systems:

- Slotted Aloha
- Carrier-Sense I as described in note #24
- Carrier-Sense II with initial random transmission delay

Total delay is the time lapse when the packet is ready for transmission until the last bit of the positive acknowledgment packet is received by the terminal. (It is necessary to include the transmission time of the acknowledgment packet in the total delay because not until the end of that transmission time is the terminal ready to transmit the next packet.)

As described in note #7, the operation is such that a terminal or a repeater transmits the packet and if an acknowledgment does not arrive within a specified period of time, it retransmits the packet. The waiting period is the time for the acknowledgment to arrive when no conflicts occur, plus a random time (when needed) for avoiding repeated conflicts. Furthermore, it is assumed that the acknowledgment packets are always correctly received with probability one. This can be accomplished in several ways. One way is to create a separate channel to handle the acknowledgment packets. No overlap occurs if sufficient bandwidth is provided, since a positive acknowledgment packet is created only when a packet is correctly received, and there must be, at most, one such packet at any given time. Another way to assure correct reception of the acknowledgment packet is to extend the transmission time of packets to include the transmission time of an eventual positive acknowledgment packet on the same channel.

The model for the delay is simplified by further assuming that the probability of the packet's success is the same whether the packet is new or has been blocked any number of times before; and this probability is assumed to be given by

$$\frac{S}{G} = \frac{\text{Throughput}}{\text{Channel Traffic}}$$

as obtained in the previous models and thus disregards the strong assumption of independency that exists in those models.* Nevertheless, for this comparative study, these simplifications appear satisfactory; and the average number of transmissions that a packet undertakes before being correctly received is given by G/S.

6.1 Notation

τ = propagation delay per hop

T = transmission time of information packet

T_b = transmission time of acknowledgment packet

S = average rate of (new) message packets offered to the channel transmission time

G = average generation rate of message packets and retransmissions offered to the channel per transmission time

R = average waiting time before retransmission is made

$\bar{X}$ = average waiting time beyond the minimum to avoid repeated conflicts.

D = E [total delay]

Let

$$\alpha = \frac{T_b}{T} \tag{20}$$

$$a = \frac{\tau}{T} \tag{21}$$

$$\delta = \frac{\bar{X}}{T} \tag{22}$$

6.2 Delay Equations

6.2.1 Aloha Channel (Note # 7) The repeater waits the transmission time and the propagation time to the next repeater, plus the transmission time of the acknowledgment packet and its propagation backward, plus, finally, a random time for avoiding repeated conflicts:

$$R = 2\tau + T + T_b + \bar{X} \tag{23}$$

*A simulation program is currently being written to validate such simplifications.

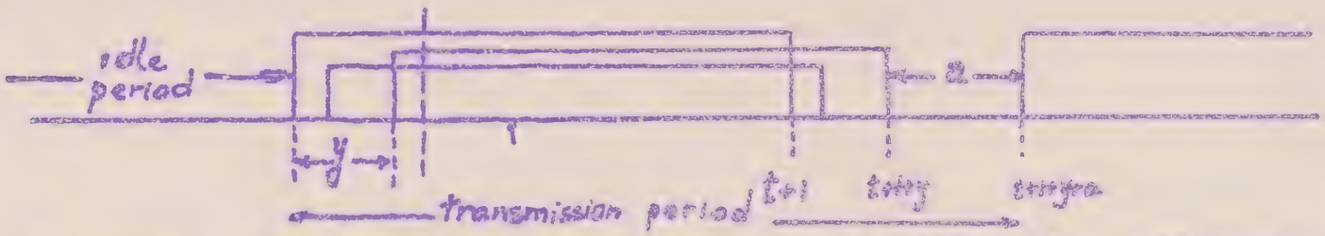
Therefore, the total delay per hop is given by

$$D = \frac{G}{s}[2\tau + T + T_b + \overline{X}] - \overline{X} \tag{24}$$

Since $G/s = e^G$, after normalization we get

$$D_{ALOHA} = e^G[2a + 1 + \alpha + \delta] - \delta \tag{25}$$

6.2.2 Carrier-Sense I (Note #24) Unlike the ALOHA channel in carrier-sense, a packet incurs an additional pre-transmission delay τ if upon its arrival, that packet detects the channel as busy.



We recall from Packet Radio Note 24 that the probability of an arbitrary packet finding the channel busy is given by

$$\frac{\overline{B} - \frac{a}{q_0}}{\overline{B} + \overline{I}}$$

where

$\overline{B}$ denotes the average busy period and is expressed by

$$\overline{B} = \frac{1 + a + \overline{y}}{q_0} \tag{26}$$

$\overline{I}$ denotes the average idle period and is given by

$$\overline{I} = \frac{1}{G} \tag{27}$$

q_0 denotes the probability that no packets accumulate at the end of a transmission period and is given by

$$q_0 = (1+aG)\, e^{-G(1+a)} \tag{28}$$

and $\overline{y}$ is given by

$$\overline{y} = a - \frac{1}{G}(1 - e^{-aG}) . \tag{29}$$

Under the condition that the packet found the channel busy, the average waiting time until the channel is detected idle (i.e., until the end of the transmission period) is, by the Poisson assumption, simply equal to $\frac{1+\bar{y}}{2}$. Therefore, the average pre-transmission delay can easily be expressed as

$$\bar{r}_1 = \frac{1+\bar{y}}{2} \cdot \Pr\{\text{the packet finds the channel busy}\}$$

$$= \frac{G(1+\bar{y})^2}{2[G(1+a+\bar{y}) + q_0]} \qquad (30)$$

Finally, the expected total delay for this system is

$$D_{C\text{-}S\ I} = \frac{G}{S}[\bar{r}_1 + 2a + 1 + \alpha + \delta] - \delta$$

with

$$\frac{G}{S} = \frac{G + 2aG - (1-e^{-aG}) + (1+aG)\,e^{-G(1+a)}}{(1+2G)(G+aG+e^{-aG})\,e^{-G(1+2a)}} \qquad (31)$$

6.23 Carrier-Sense II (with initial random transmission delay) Similar to Carrier-Sense I, a packet in this scheme incurs an initial delay which we denote by r_2. In order to compute $\bar{r}_2$, one has to consider the following situations:

- An arbitrary packet, upon arrival, will find the channel idle with probability $\frac{\bar{I}}{\bar{B}+\bar{I}}$, in such case, its initial wait is $\bar{t}_1$.
- An arbitrary packet, upon arrival, will find the channel in the in the first t_1 seconds of a busy period with probability $\frac{\bar{t}_1}{\bar{B}+\bar{I}}$. In such case, its initial delay is $\frac{\bar{t}_1}{2}$.
- An arbitrary packet, upon arrival, will find the channel in the remaining part of a busy period with probability $\frac{\bar{B}-\bar{t}_1}{\bar{B}+\bar{I}}$, in which case the initial wait is $\frac{T+1+\bar{t}}{2}$.

Therefore

$$\bar{r}_2 = \frac{\bar{I}}{\bar{B} + \bar{I}} \bar{t}_1 + \frac{\bar{t}_1}{\bar{B} + \bar{I}} \frac{\bar{t}_1}{2} + \frac{\bar{B} - \bar{t}_1}{\bar{B} + \bar{I}} \cdot \frac{\bar{T} + 1 + \bar{t}}{2} \tag{32}$$

where $\bar{t}_1$, $\bar{t}$, $\bar{B}$, and $\bar{I}$ are given by Equations 5, 6, 16, and 18 respectively.

Unlike the previous two schemes, this design requires no waiting time $\bar{X}$ beyond the minimum to avoid repeated conflicts; therefore, the expected total delay is expressed by

$$D_{C\text{-}S\ II} = \frac{G}{s} [\ \bar{r}_2 + 2 + T + T_b] \tag{33}$$

and after normalizing, we get

$$D_{C\text{-}S\ II} = \frac{G}{s} [\ a\bar{r}_2 + 2a + 1 + \alpha] \tag{34}$$

6.3 Results

Figures 6, 7, and 8* display the tradeoff between delay and throughput for different values of p, and three values of a: 0.01, 0.05, and 0.1. It is to be noted that the interesting part of these curves is the lower envelope which gives the maximum throughput for a given delay, or the minimum delay for a given throughput.

Figure 9* is a comparison (for values a = 0.01, 0.05, and 0.1) of the three systems:

- Slotted Aloha
- Carrier-Sense I (note #24)
- Carrier-Sense II (this note)

in terms of the tradeoff between throughput and delay. For Carrier-Sense II, only the lower envelopes are plotted.

We conclude that the Slotted Aloha system is rather insensitive to the value of a, and that, for the useful range of a ($0 < a < 0.1$), Carrier Sense II (with initial random transmission delay) is by far superior to the two other systems.

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*All curves shown are for the parameters $\alpha = 1$ and $\delta = 2$ (as used in Note #7)

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7 X 10 INCHES MADE IN U.S.A.
KEUFFEL & ESSER CO.

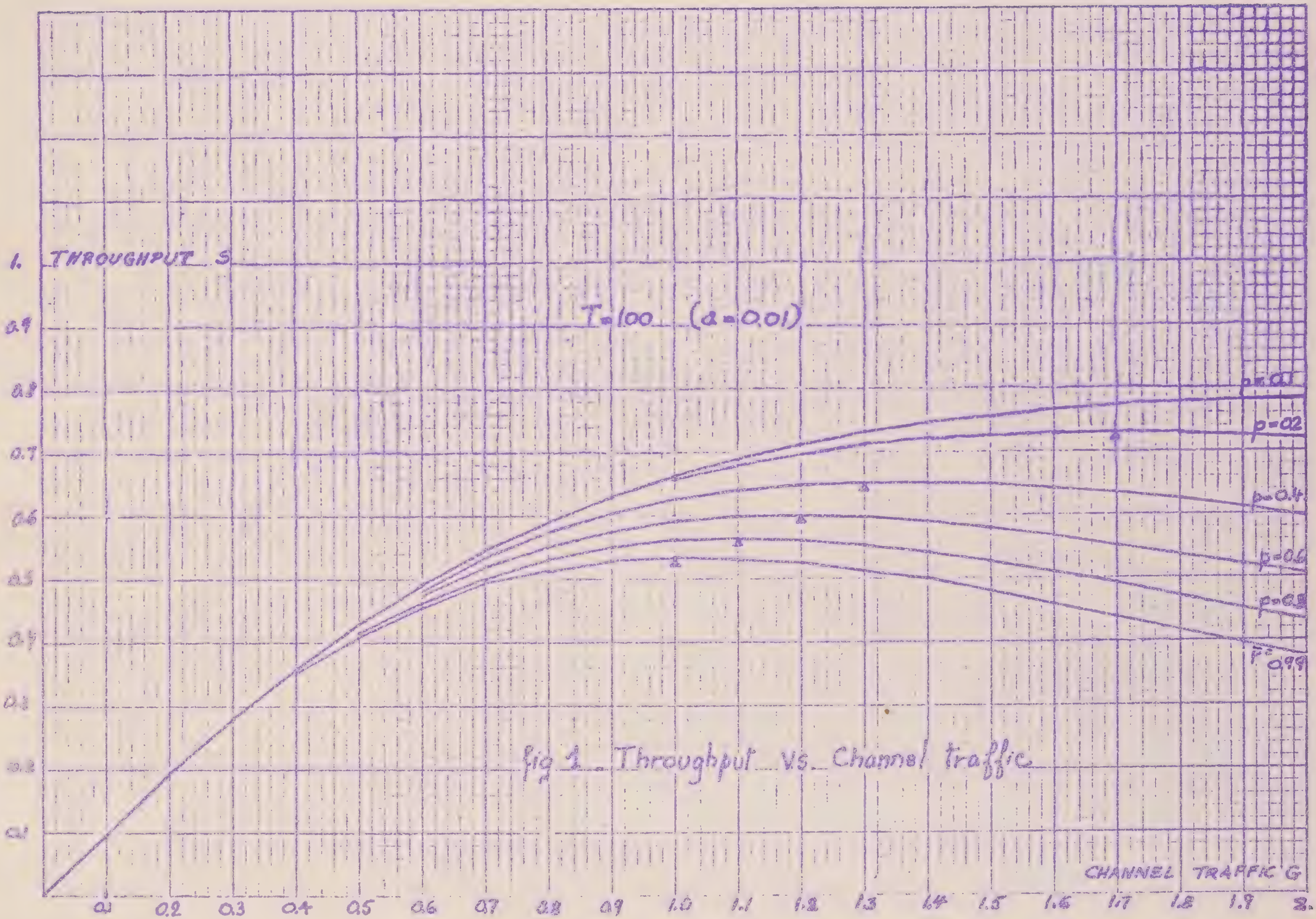

fig 1 Throughput Vs Channel traffic

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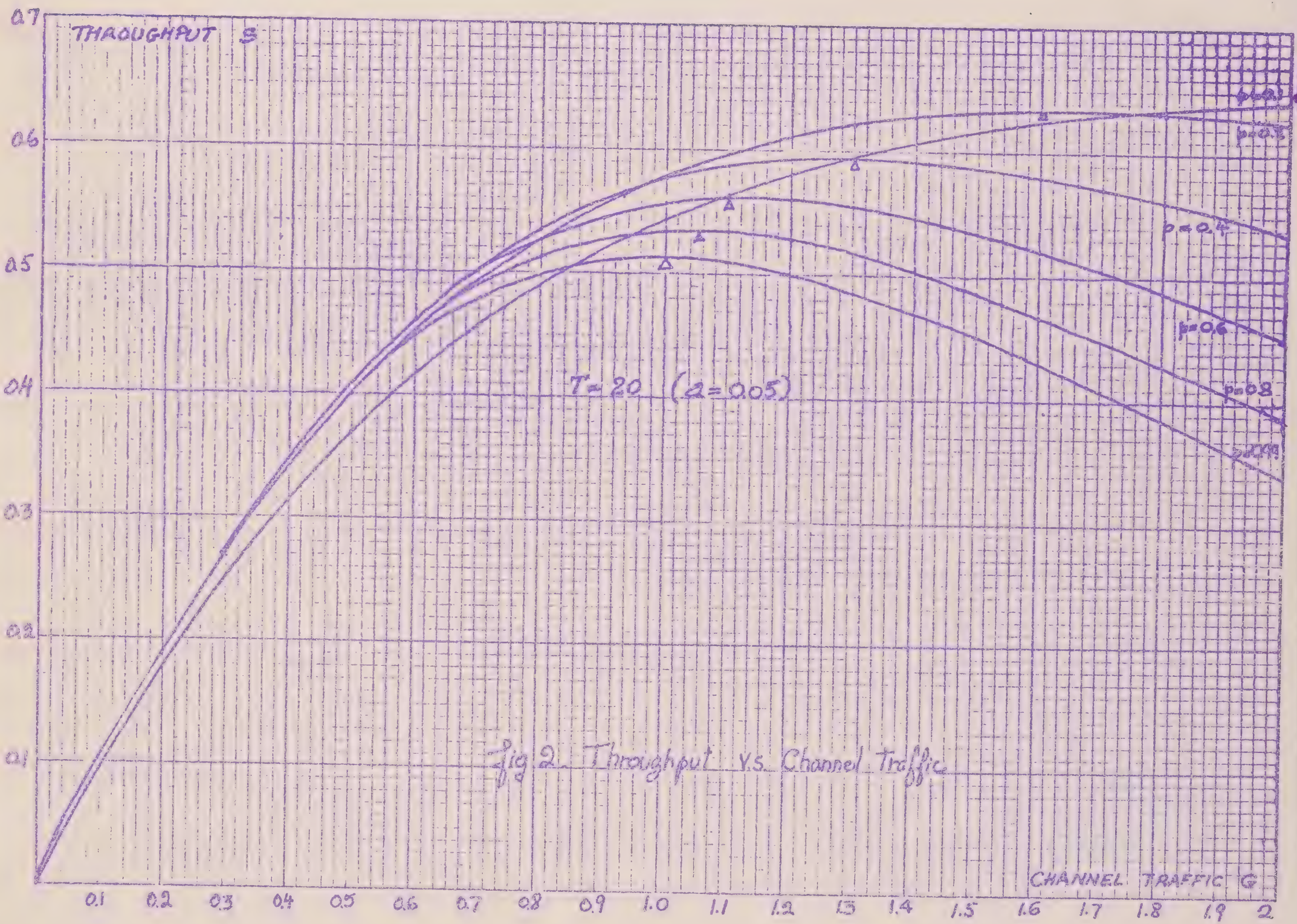
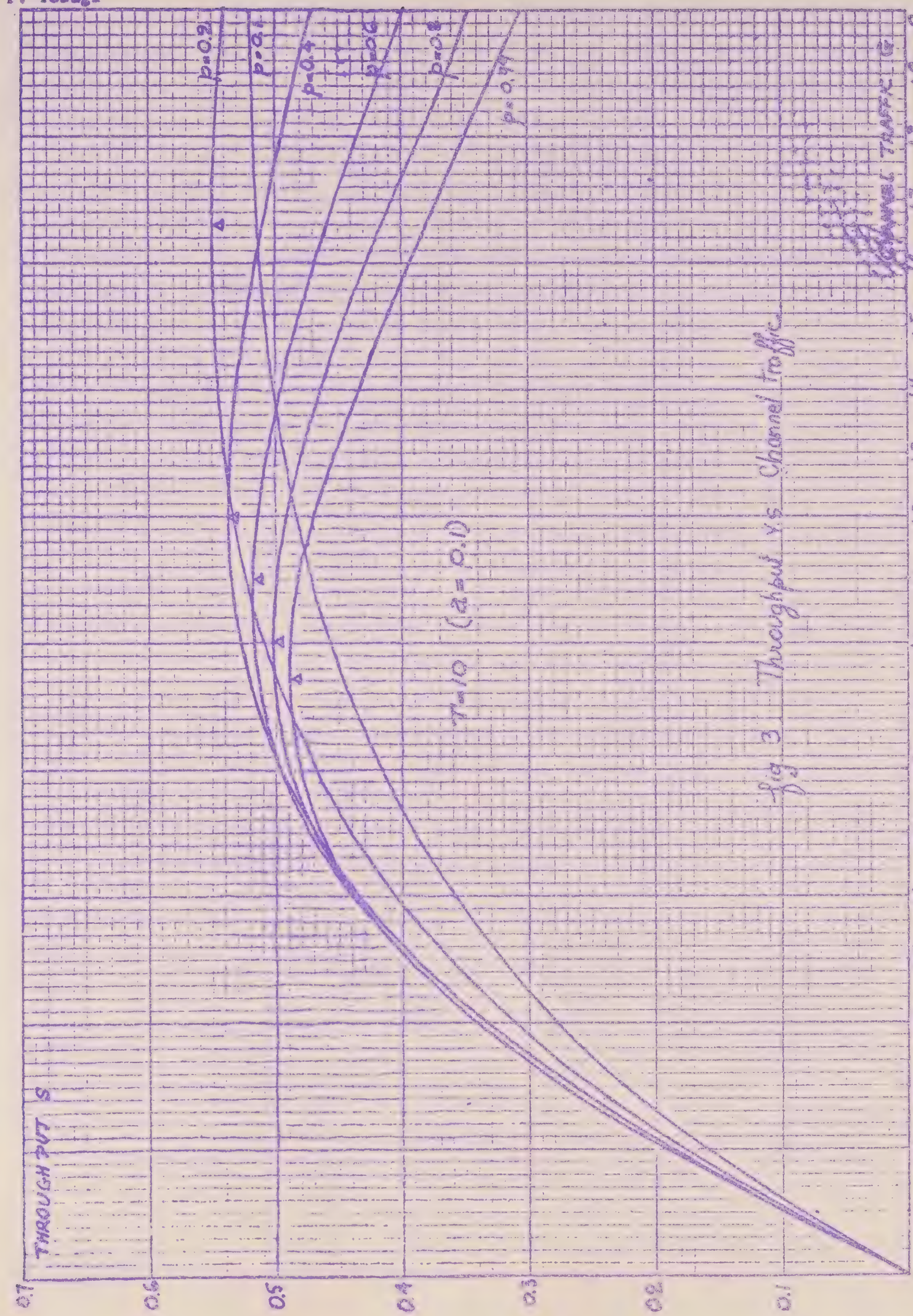


Fig 2 Throughput V.S. Channel Traffic


p=0.9
p=0.8
p=0.7
p=0.6
p=0.5
p=0.4
THROUGH PUT S
0.7
0.6
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1.9
2
Channel Traffic G
T=10 (a=0.1)
Fig 3 Throughput vs Channel traffic

K&E SEMI-LOGARITHMIC 46 4972
2 CYCLES X 70 DIVISIONS MADE IN U.S.A.
KEUFFEL & ESSER CO.

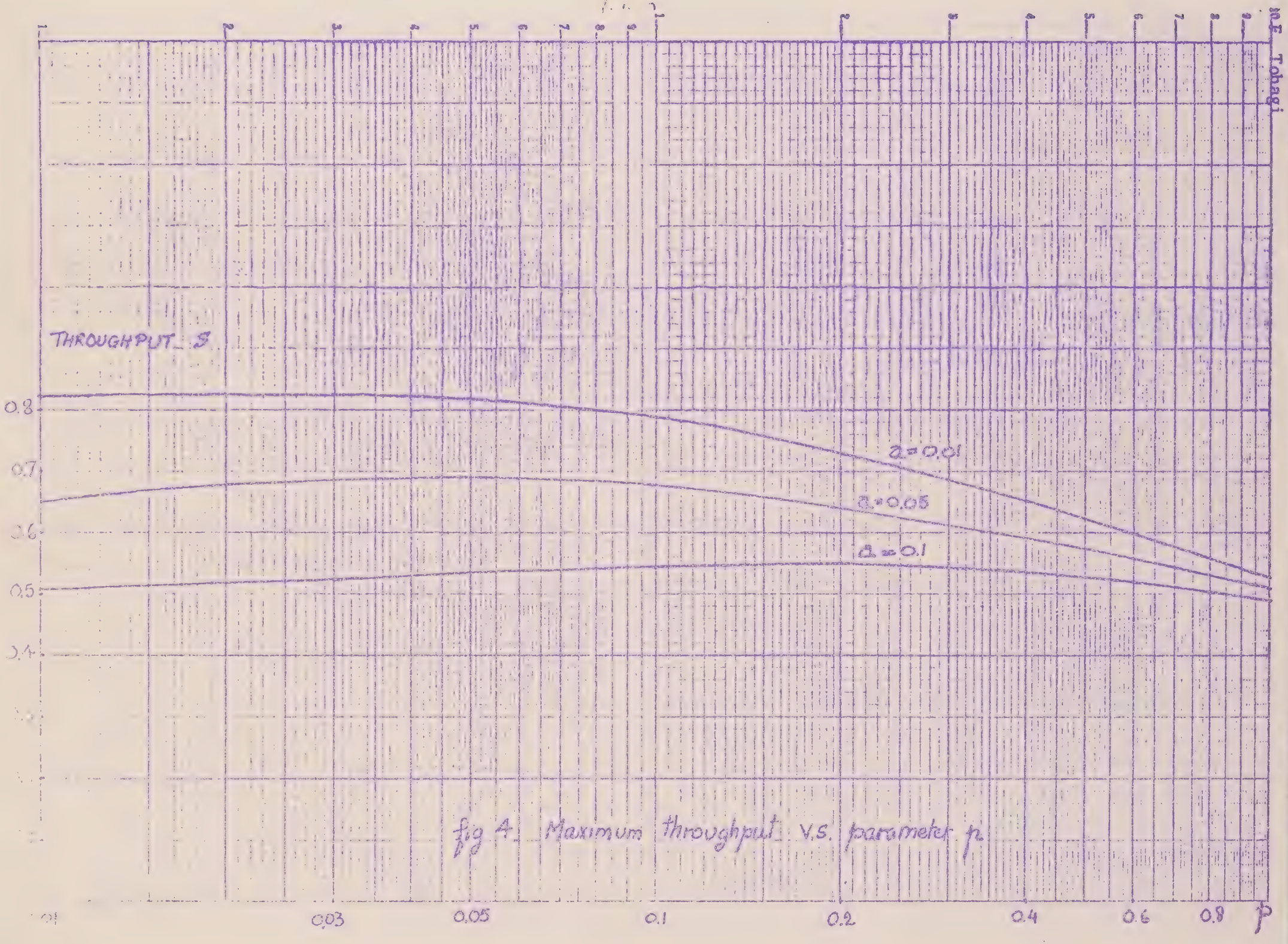

fig 4 Maximum throughput V.S. parameter p.

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K&E 5 X 5 TO ½ INCH 7 X 10 INCHES 46 0860 MADE IN U.S.A.
KEUFFEL & ESSER CO.

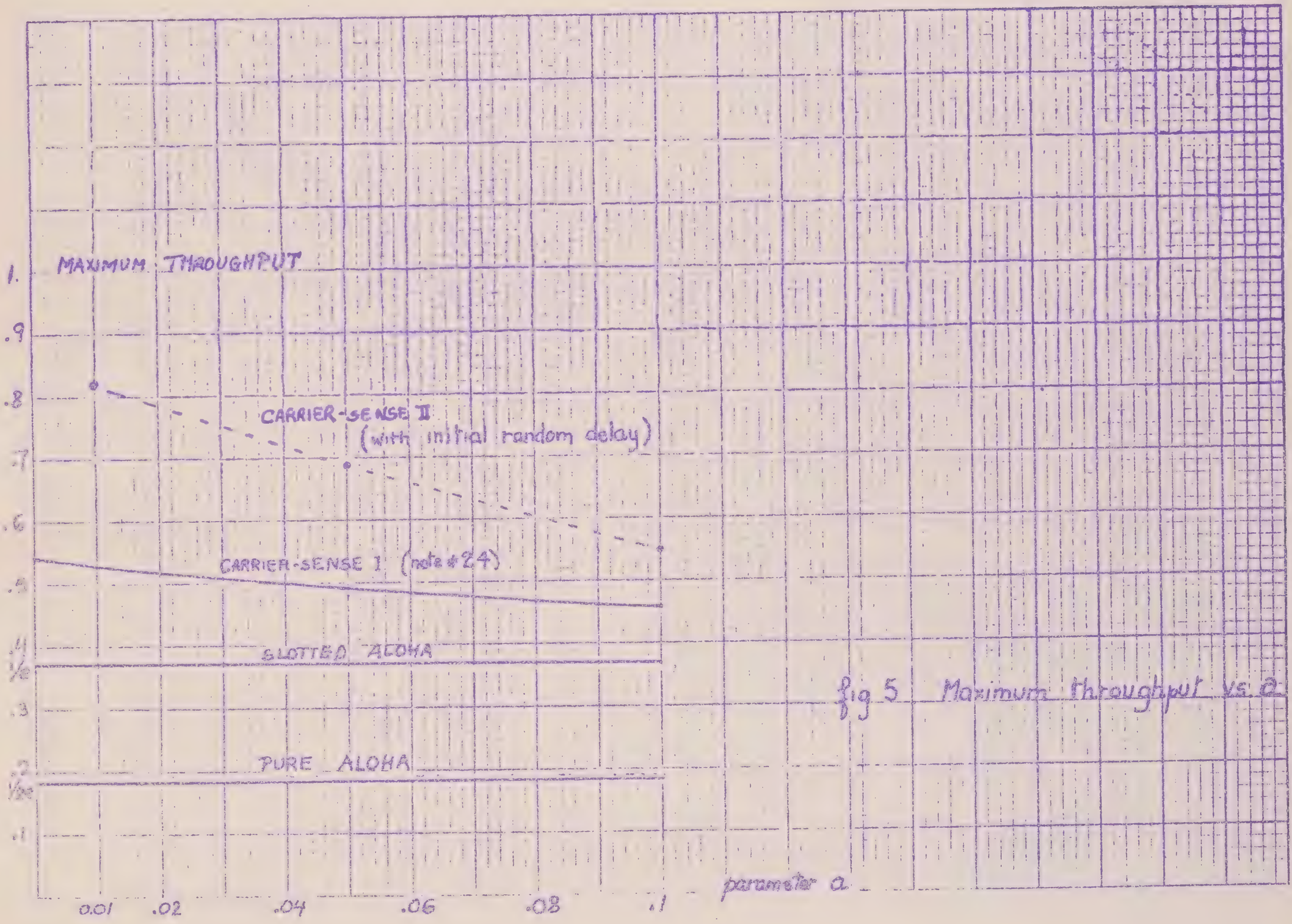


fig 5 Maximum throughput vs. a

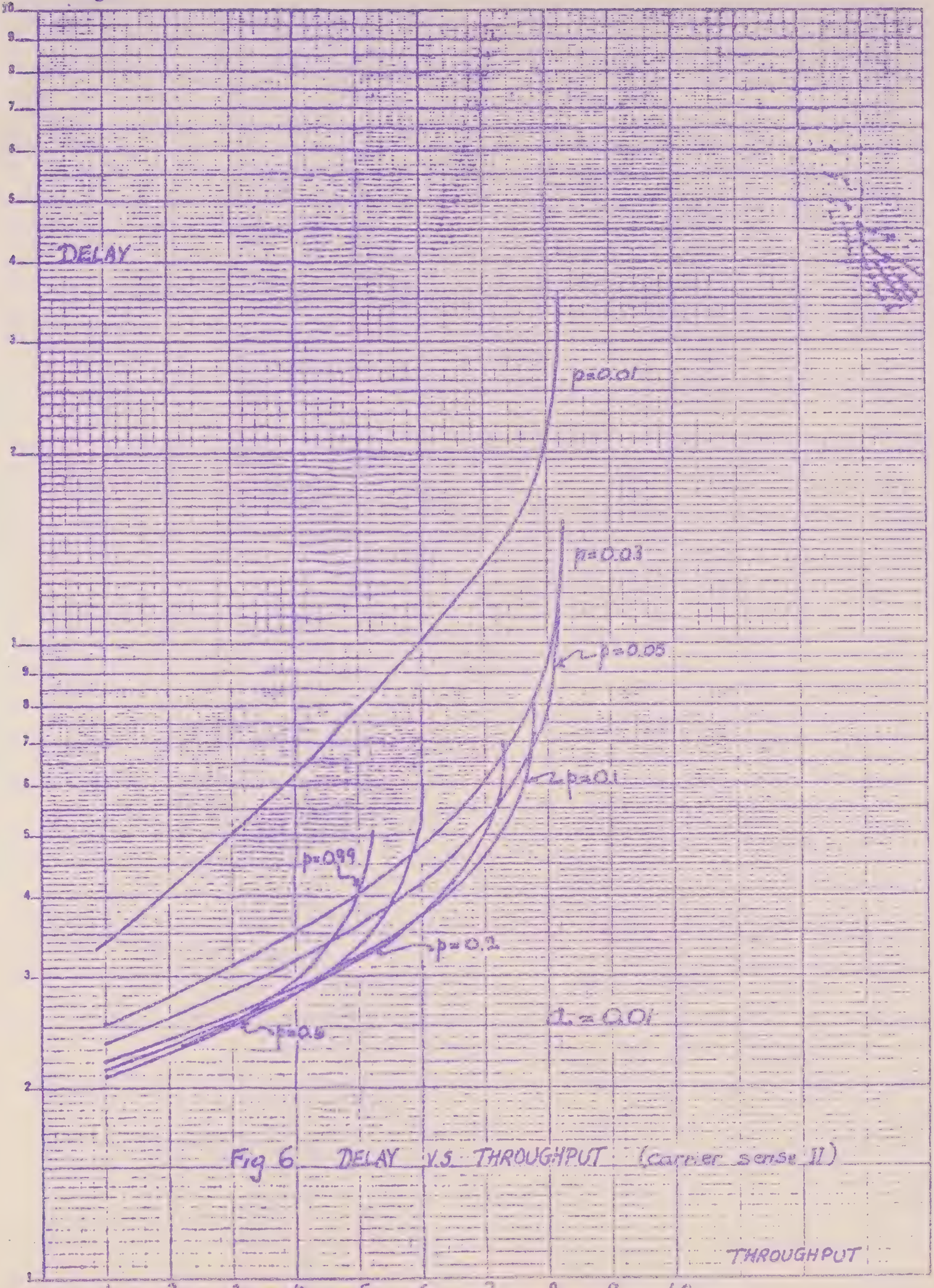


Fig 6 DELAY V.S. THROUGHPUT (carrier sense II)

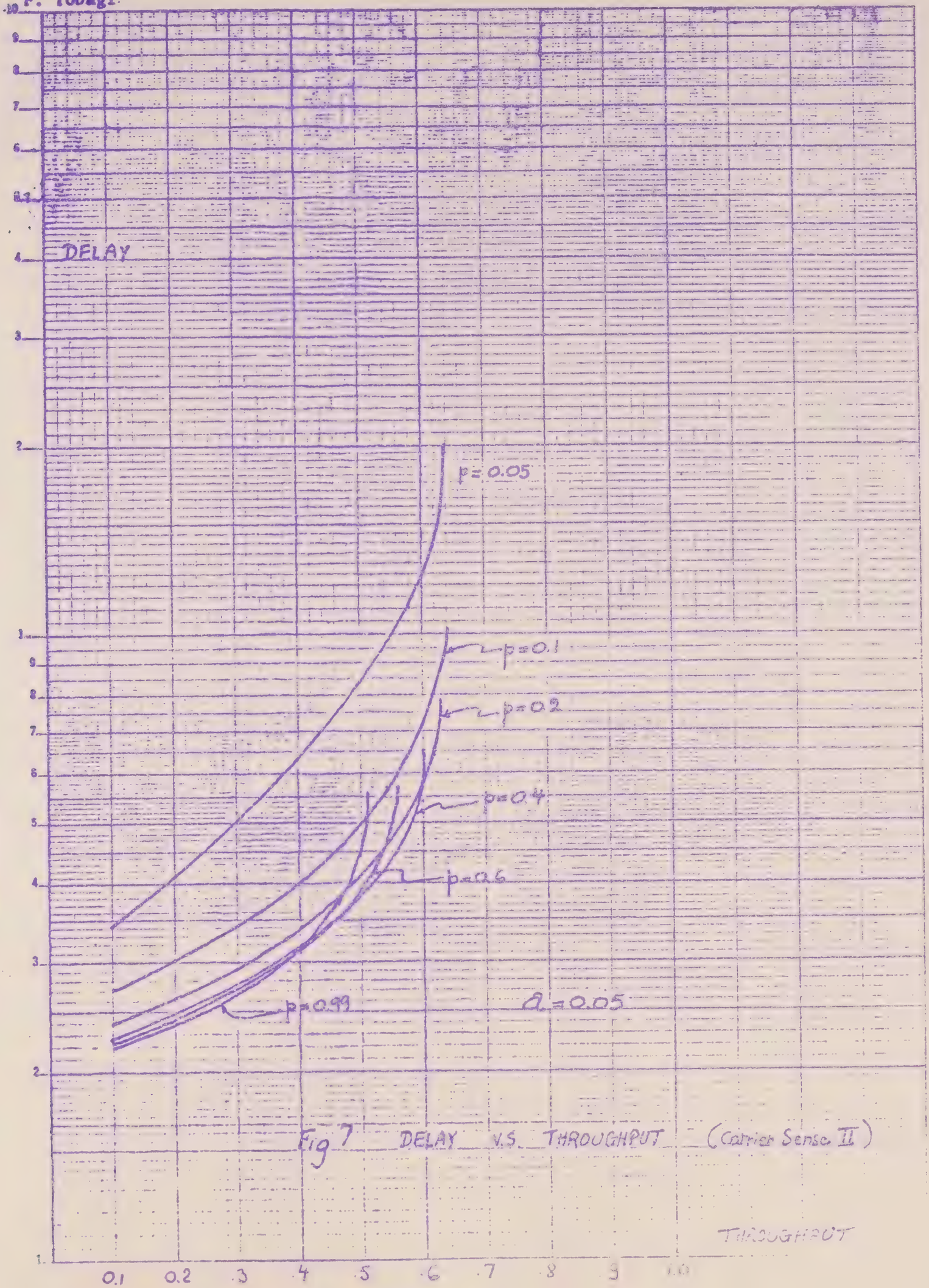


Fig 7 DELAY V.S. THROUGHPUT (Carrier Sense II)

KEUFFEL & ESSER CO. 48 4972
SEMI-LOGARITHMIC
2 CYCLES X 70 DIVISIONS

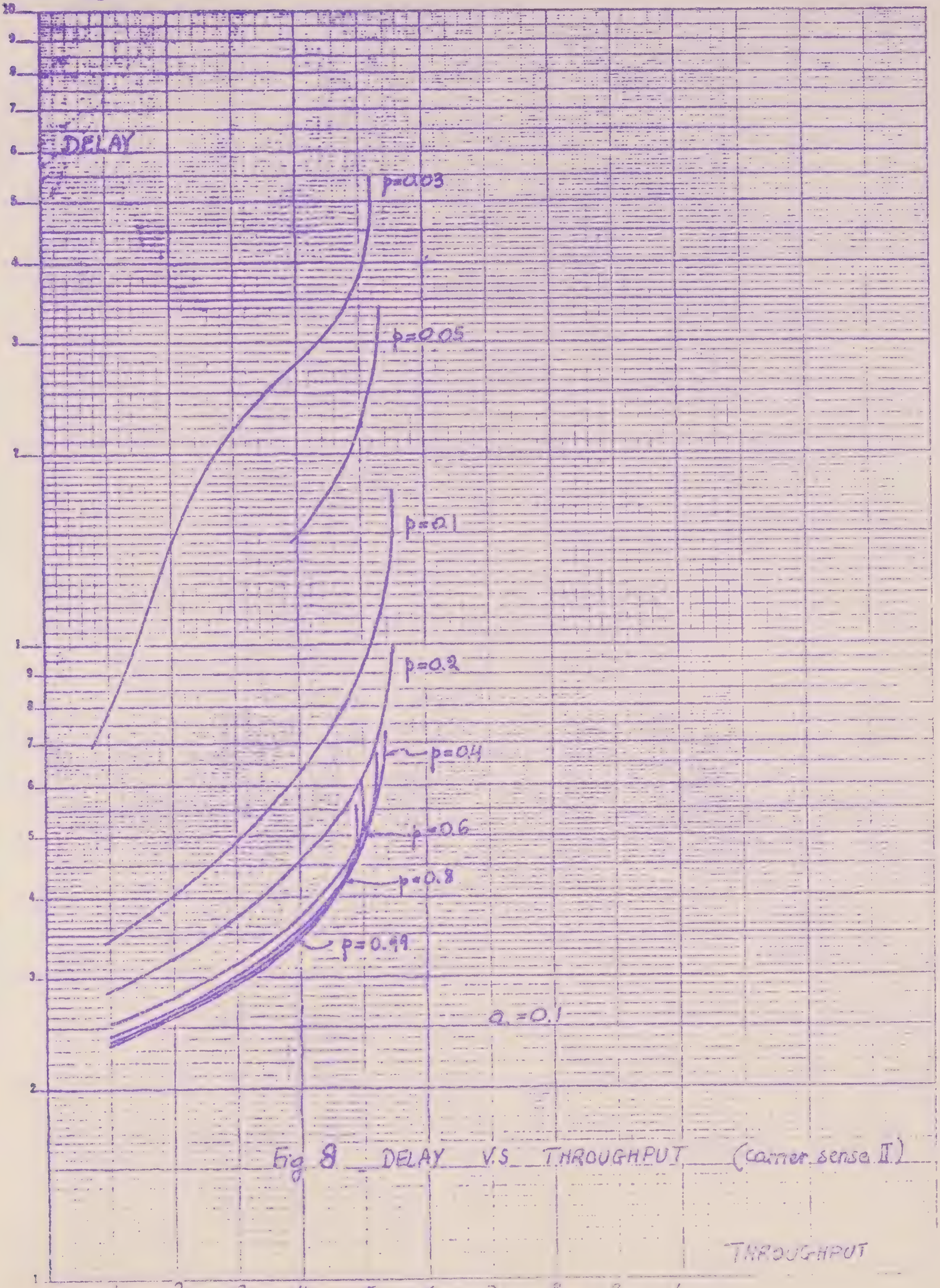
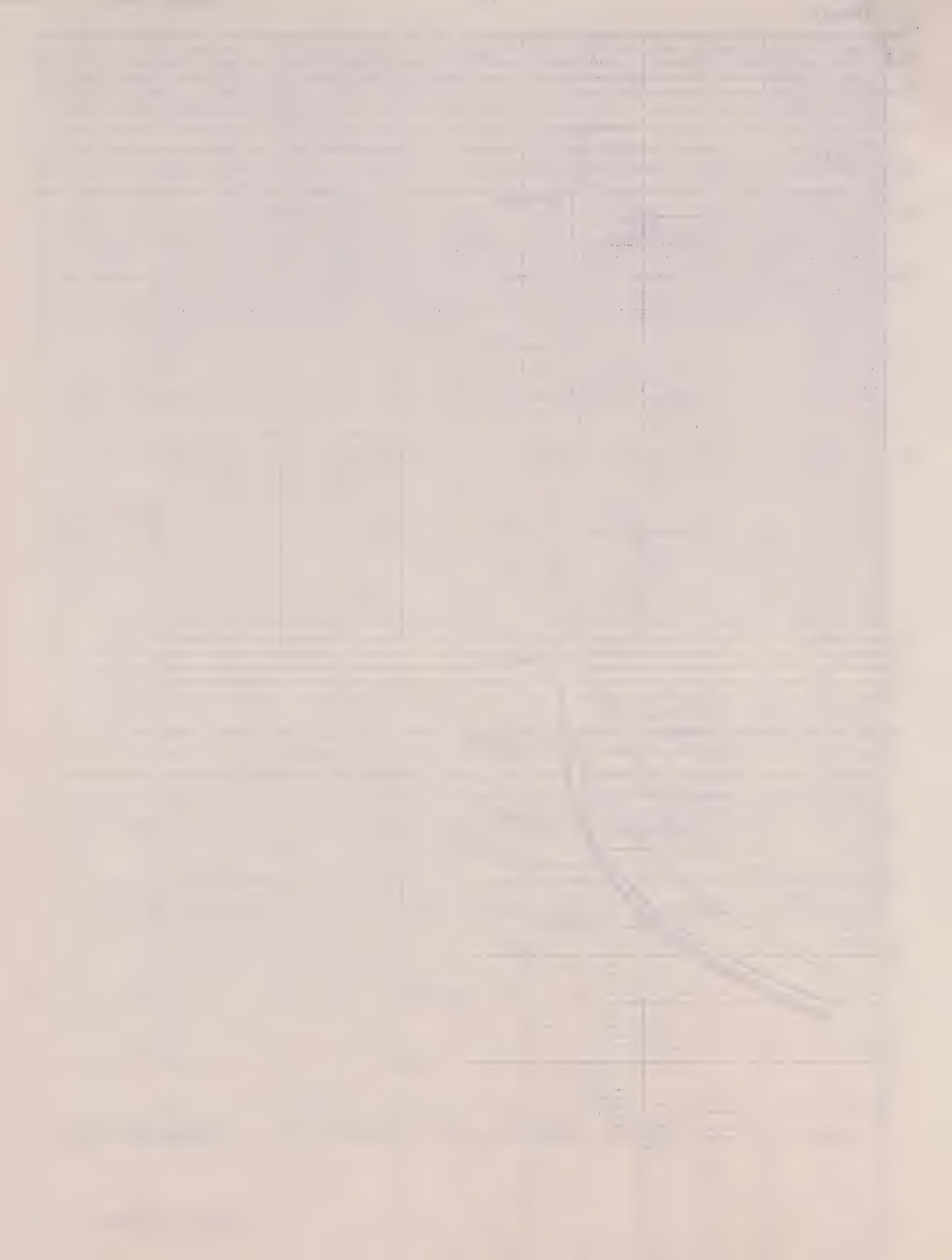


Fig 8 DELAY V.S THROUGHPUT (carrier sense II)

KEUFFEL & ESSER CO. SEMI-LOGARITHMIC 46 4972 2 CYCLES X 70 DIVISIONS



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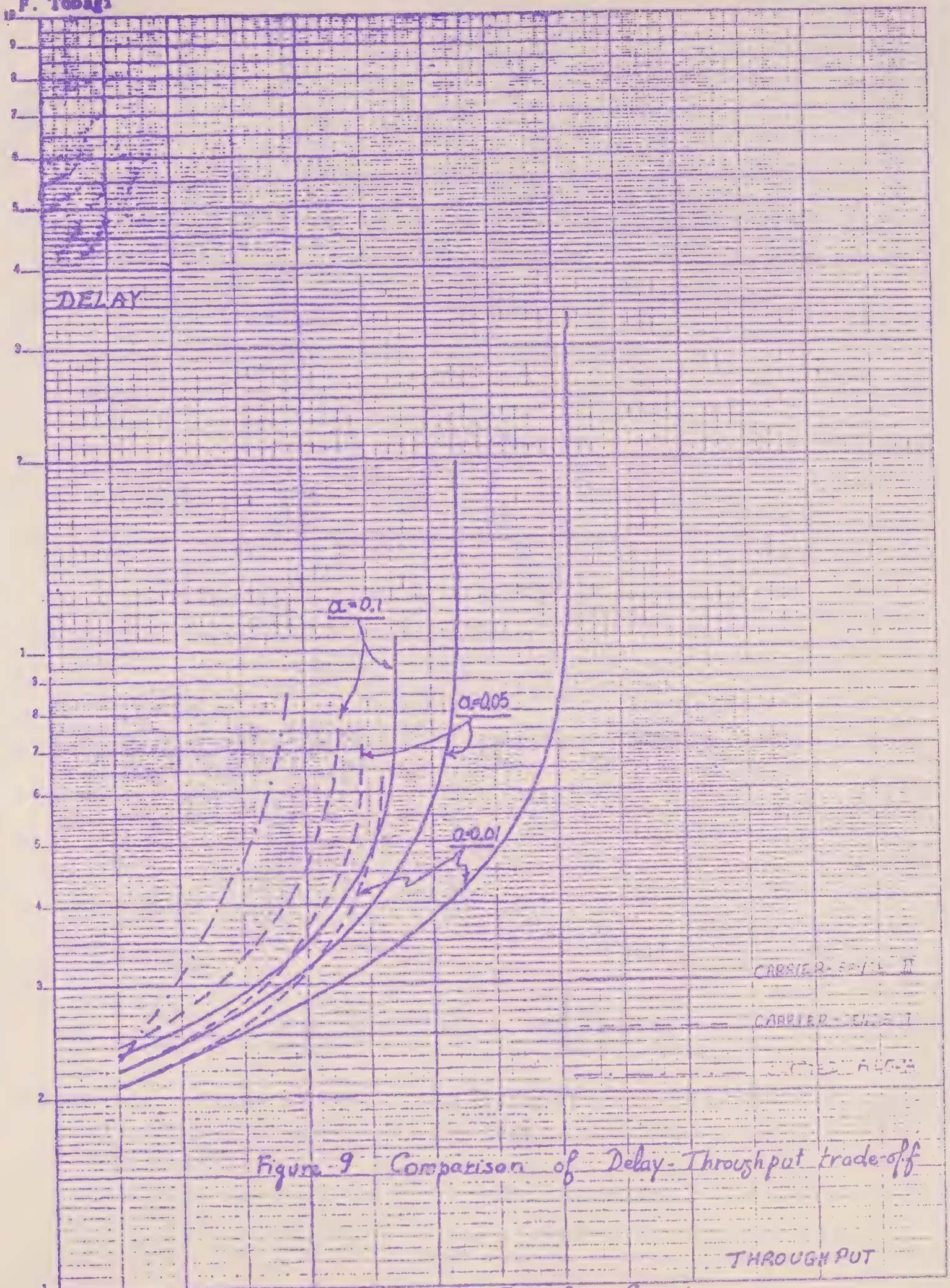

Figure 9 Comparison of Delay-Throughput trade-off

APPENDIX

In this appendix we show that, at the expense of some approximations, we can get a closed form solution for the throughput when p has small values ($p < 0.1$). The validation of these approximations is done by comparing the results obtained in this appendix to those obtained from the PL/I program, for $p = 0.1$.

Distribution of Idle Time Between Two Transmission Periods

From equation 2 we have

$$\Pr\{t_N < k\} = q^{(k+1)N} \; e^{g\left(\frac{q(1-q^k)}{p} - k\right)} \qquad (a.1)$$

When p is small we can make the following approximation:

$$q^k = (1-p)^k \simeq 1 - kp \qquad (a.2)$$

and therefore, we can rewrite Equation a.1 as

$$\Pr\{t_N > k\} \simeq q^{(k+1)N} \; e^{-kpg} = q^N \left[q^N e^{-pg}\right]^k \qquad (a.3)$$

Let $t^*_{N>}(z)$ and $t^*_N(z)$ be the generating functions defined by

$$t^*_{N>}(z) = \sum_{k=0}^{\infty} \Pr\{t_N > k\} z^k \qquad (a.4)$$

$$t^*_N(z) = \sum_{k=0}^{\infty} \Pr\{t_N = k\} z^k \qquad (a.5)$$

We have

$$t^*_{N>}(z) = q^N \sum_{k=0}^{\infty} \left(q^N e^{-pg} z\right)^k = \frac{q^N}{1 - q^N e^{-pg} z} \qquad (a.6)$$

Since

$$\Pr\{t_N = k\} = \Pr\{t_N > k-1\} - \Pr\{t_N > k\} \qquad k > 0$$

and $\Pr\{t_N = 0\} = 1 - \Pr\{t_N > 0\}$

we have

$$t_N^*(z) = 1 + t_{N>}^*(z)(z-1) = 1 + \frac{q^N(z-1)}{1 - q^N e^{-pg} z} \qquad (a.7)$$

The averages defined in Equation 5 can now be written as

$$\bar{t}_N = \frac{\partial t_N^*(z)}{\partial z}\bigg|_{z=1} = \frac{q^N}{1 - q^N e^{-pg}} \qquad (a.8)$$

and in particular

$$\bar{t}_1 = \frac{q}{1 - q e^{-pg}} \qquad (a.9)$$

Equation 6 which defines $\bar{t}$ as $\sum_{N=1}^{\infty} \bar{t}_N \frac{P_N}{1-P_0}$ does not lead to a closed form expression. Instead, we replace $\bar{t}$ by $\bar{t}'$ which is defined as

$$\bar{t}' = \frac{C}{1 - C e^{-pg}} \qquad (a.10)$$

where $C = \sum_{N=1}^{\infty} q^N \frac{P_N}{1-P_0}$. ($\bar{t}'$ is smaller than $\bar{t}$ since $\bar{t}_N = \frac{q^N}{1-q^N e^{-pg}}$ is a convex function of q^N).

C can be expressed as

$$C = \frac{e^{-(T+1)pg} - P_0}{1 - P_0} = \frac{P_0^p - P_0}{1 - P_0} \qquad (a.11)$$

and therefore

$$\bar{t}' = \frac{P_0^p - P_0}{1 - P_0 - (P_0^p - P_0)e^{-pg}} \qquad (a.12)$$

Probability of Success over a Transmission Period

Define the following generating functions:

$$L_N^*(z) = \sum_{\ell=N-1}^{\infty} \Pr\{L_N = \ell\} z^{\ell} \qquad (a.13)$$

$$L_N^*(z/k) = \sum_{\ell=N-1}^{\infty} \Pr\{L_N = \ell / t_N = k\}\, z^{\ell} \qquad (a.14)$$

It is clear that

$$L_N^*(z/k) = e^{kg(z-1)}\, z^{N-1} \qquad (a.15)$$

Removing the condition on k we get

$$L_N^*(z) = \sum_{k=0}^{\infty} L_N^*(z/k) \cdot \Pr\{t_N = k\}$$

$$= z^{N-1} \sum_{k=0}^{\infty} e^{kg(z-1)} \cdot \Pr\{t_N = k\}$$

$$= z^{N-1}\, t_N^*(e^{g(z-1)})$$

$$= \frac{q^N (e^{g(z-1)} - 1)\, z^{N-1}}{1 - q^N e^{-pg} e^{g(z-1)}} + z^{N-1} \qquad (a.16)$$

The probability of success $P_s(N)$ defined in Equation 9 is now simply expressed as

$$P_s(N) = L_N^*(q)$$

$$= q^{N-1} - \frac{(1-e^{-gp}) q^{2N-1}}{1 - q^N e^{-2gp}} \qquad (a.17)$$

In particular

$$P_s(1) = 1 - \frac{(1-e^{-gp})q}{1 - q\, e^{-2gp}} \qquad (a.18)$$

Here again Equation 10 defines

$$P_s = \sum_{N=1}^{\infty} P_s(N) \cdot \frac{P_N}{1 - P_0} \qquad (a.19)$$

which does not lead to a closed form expression. Instead, we replace P_s by P_s' which is defined as

$$P_s' = C - \frac{(1-e^{-gp})C'}{q(1-C\,e^{-gp})} \qquad (a.20)$$

where

$$C = \sum_{N=1}^{\infty} q^N \frac{P_N}{1-P_0} = \frac{P_0^p - P_0}{1-P_0} \qquad (a.21)$$

and

$$C' = \sum_{N=1}^{\infty} q^{2N} \frac{P_N}{1-P_0} = \frac{P_0^{1-q^2} - P_0}{1-P_0} \qquad (a.22)$$

Finally P_s' can be expressed as shown in the following equation:

$$P_s' = \frac{P_0^p - P_0}{q\left(1-P_0\right)} - \frac{\left(1-e^{-gp}\right)\left(P_0^{1-q^2} - P_0\right)}{q\left(1-P_0\right) - q\,e^{-2gp}\left(P_0^p - P_0\right)} \qquad (a.23)$$

Knowing that $P_0 = e^{-G(1+a)}$, substituting equations (a.9), (a.12), (a.18) and (a.22) into Equation 19 provides us with a closed form solution for $s(G,p,a)$ when p is small.

In Table 1, we compare, for $p = 0.1$, the "exact" results obtained by the PL/I program to those obtained by the approximation method, and note that the closed form solution is quite satisfactory for $p < 0.1$.

G \ a	0.01		0.05		0.1	
	Exact	Approx.	Exact	Approx.	Exact	Approx.
0.1	0.098	0.098	0.095	0.094	0.091	0.090
0.2	0.192	0.192	0.179	0.178	0.166	0.164
0.3	0.279	0.279	0.252	0.251	0.227	0.223
0.4	0.358	0.358	0.316	0.314	0.279	0.272
0.5	0.428	0.428	0.370	0.367	0.322	0.313
0.6	0.490	0.490	0.417	0.413	0.358	0.347
0.7	0.544	0.544	0.457	0.453	0.389	0.376
0.8	0.589	0.590	0.490	0.486	0.415	0.400
0.9	0.628	0.630	0.519	0.515	0.437	0.421
1.0	0.661	0.663	0.543	0.539	0.456	0.440
1.1	0.689	0.691	0.563	0.560	0.472	0.456
1.2	0.711	0.714	0.580	0.578	0.486	0.470
1.3	0.730	0.733	0.594	0.593		
1.4	0.745	0.749	0.606	0.605		
1.5	0.757	0.761	0.616	0.616		
1.6	0.766	0.771	0.624	0.625		
1.7	0.773	0.778	0.630	0.632		
1.8	0.778	0.784	0.635	0.638		
1.9	0.781	0.787	0.639	0.643		
2.0	0.783	0.790	0.642	0.647		

TABLE 1

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